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Upward Extension of the Jacobi Matrix for Orthogonal Polynomials

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Orthogonal polynomials on the real line always satisfy a three-term recurrence relation. The recurrence coefficients determine a tridiagonal semi-infinite matrix (Jacobi matrix) which uniquely characterizes the orthogonal polynomials. We investigate new orthogonal polynomials by adding to the Jacobi matrix r new rows and columns, so that the original Jacobi matrix is shifted downward. The r new rows and columns contain $2r$ new parameters and the newly obtained orthogonal polynomials thus correspond to an upward extension of the Jacobi matrix. We give an explicit expression of the new orthogonal polynomials in terms of the original orthogonal polynomials, their associated polynomials, and the $2r$ new parameters, and we give a fourth order differential equation for these new polynomials when the original orthogonal polynomials are classical. Furthermore we show how the orthogonalizing measure for these new orthogonal polynomials can be obtained and work out the details for a one-parameter family of Jacobi polynomials for which the associated polynomials are again Jacobi polynomials.

1. INTRODUCTION

The construction of families of orthogonal polynomials on the real line from a given system of orthogonal polynomials (or from a given

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orthogonalizing weight μ) has been the subject of various investigations [3, 4, 6, 7, 9, 11, 14–16, 25]. Let P_n ($n=0, 1, 2, ...$) be a sequence of monic orthogonal polynomials on the real line, with orthogonality measure μ, then these polynomials satisfy a three-term recurrence relation

$$P_{n+1}(x)=(x-b_n)\,P_n(x)-a_n^2 P_{n-1}(x), \qquad n\geqslant 0, \tag{1.1}$$

with $b_n \in \mathbb{R}$ and $a_n^2>0$ and initial conditions $P_0=1$, $P_{-1}=0$. The corresponding orthonormal polynomials are

$$p_n(x)=\frac{1}{a_1 a_2 \cdots a_n} P_n(x),$$

and they satisfy the three-term recurrence relation

$$xp_n(x)=a_{n+1}p_{n+1}(x)+b_n p_n(x)+a_n p_{n-1}(x), \qquad n\geqslant 0. \tag{1.2}$$

Putting the recurrence coefficients in an infinite tridiagonal matrix gives the Jacobi matrix

$$J=\begin{pmatrix} b_0 & 1 & & & \\ a_1^2 & b_1 & 1 & & \\ & a_2^2 & b_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

or the symmetric Jacobi matrix

$$J_s=\begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Interesting new Jacobi matrices can be obtained by deleting the first r rows and columns, and the corresponding orthogonal polynomials are then the associated polynomials of order r, denoted by $p_n^{(r)}$. Instead of deleting rows and columns we will add r new rows and columns at the beginning of the Jacobi matrix, thus introducing $2r$ new parameters. We call the corresponding new orthogonal polynomials anti-associated of order r and denote them by $p_n^{(-r)}$. In Section 4 we will give explicit formulas for these anti-associated polynomials in terms of the original orthogonal polynomials and the associated polynomials. We also give a fourth order differential equation in case the original orthogonal polynomials are classical. The construction of the orthogonality measure for the anti-associated polynomials is done in Section 5. The analysis requires the knowledge of the orthogonality measure of the original system and of the associated polynomials, and also the asymptotic behavior of both P_n and $P_n^{(1)}$ as $n \to \infty$. These things are known for a particular one-parameter family of Jacobi polynomials which we will call Grosjean polynomials (after Grosjean who gave a number of interesting properties of this family in [9]) and which are considered in Section 3.

The construction of new orthogonal polynomials by changing and shifting recurrence coefficients has been under investigation by others. If the original system of orthogonal polynomials has constant recurrence coefficients, then one essentially deals with Chebyshev polynomials of the second kind. Changing a finite number of the recurrence coefficients leads to Bernstein–Szegő weights [22, Sect. 2.6], i.e., weights of the form $\sqrt{1-x^2}/\rho(x)$, where ρ is a positive polynomial on the orthogonality interval $[-1, 1]$. Such finite perturbations have been considered among others by Geronimus [6], Grosjean [7], Sansigre and Valent [21], and Zabelin [25]. The limiting case, when the number of changed recurrence coefficients tends to infinity (with changes becoming smaller) is treated by Geronimo and Case [4], Dombrowski and Nevai [3], and in [24]. Instead of starting with constant recurrence coefficients one can also start with periodic recurrence coefficients. Finite perturbations of periodic recurrence coefficients are considered by Geronimus [6], Grosjean [8] and the limiting case by Geronimo and Van Assche [5]. Changing a finite number of recurrence coefficients of a general system of orthogonal polynomials has been investigated by Marcellán et al. [11] and the limiting case by Nevai and Van Assche [15]. Associated polynomials correspond to deleting rows and columns of the Jacobi matrix, or equivalently to a positive shift in the recurrence coefficients. This situation is rather well known since it corresponds to numerator polynomials in Padé approximation. A general study of such a shift in the recurrence coefficients can be found in Belmehdi [1] or Van Assche [23], and for Jacobi polynomials we refer to Grosjean [9] and Lewanowicz [10]. The case where a shift in the recurrence coefficients is made together with a change of a finite number of coefficients is treated by Nevai [14] and Peherstorfer [16]. In particular, Nevai shows that when the original system is orthogonal on an interval Δ with weight function w, then the orthogonal polynomials with weight function $w(x)/|f_+(x)|^2$ on Δ, where $f(z) = a + S(Bw, z)$ and $f_+(x) = \lim_{\varepsilon \to 0+} f(x + i\varepsilon)$, with $S(Bw, z)$ the Stieltjes transform of the function Bw and B a polynomial of degree l, have recurrence coefficients which can be obtained by shifting the original recurrence coefficients and changing a finite number of the initial recurrence coefficients. Note that this generalizes the Bernstein–Szegő polynomials.

2. BACKGROUND AND NOTATION

Let us consider the family of monic orthogonal polynomials P_n, $n=0, 1, 2, ...$, where P_n has degree n, defined by the three-term recurrence relation

$$P_{n+1}(x)=(x-b_n)\,P_n(x)-a_n^2P_{n-1}(x), \qquad n\geqslant 0, \quad a_n^2\neq 0, \tag{2.1}$$

with $P_{-1}=0$ and $P_0=1$. The sequences b_n $(n\geqslant 0)$ and a_n^2 $(n\geqslant 1)$ also generate the *associated* monic polynomials of order r (r a positive integer) $P_n^{(r)}$, $n=0, 1, 2, ...$, by the shifted recurrence

$$P_{n+1}^{(r)}(x)=(x-b_{n+r})\,P_n^{(r)}(x)-a_{n+r}^2P_{n-1}^{(r)}(x), \qquad n\geqslant 0,$$

with $P_{-1}^{(r)}=0$ and $P_0^{(r)}=1$. The recurrence relation (2.1) can also be written in operator form using the non-symmetric Jacobi matrix J given by

$$J=\left(\begin{array}{cccc|ccc}
b_0 & 1 & & & & & \\
a_1^2 & b_1 & 1 & & & & \\
 & a_2^2 & b_2 & 1 & & & \\
 & & \ddots & \ddots & \ddots & & \\ \hline
 & & & a_r^2 & b_r & 1 & \\
 & & & & a_{r+1}^2 & b_{r+1} & 1 \\
 & & & & & \ddots & \ddots \quad \ddots
\end{array}\right),$$

as

$$x\vec{P}=J\vec{P},$$

where $\vec{P}=(P_0, P_1, P_2, ...)^t$. In the same way the recurrence relation defining the associated polynomials of order r is represented by

$$x\vec{P}^{(r)}=J^{(r)}\vec{P}^{(r)},$$

where $\vec{P}^{(r)}=(P_0^{(r)}, P_1^{(r)}, P_2^{(r)}, ...)^t$ and $J^{(r)}$ is the Jacobi matrix obtained by deleting the first r rows and columns of J.

Observe that $J^{(r)}=J$ for every integer r if and only if $b_n=b$ for $n\geqslant 0$ and $a_n^2=a^2$ for $n>0$. The corresponding family of orthogonal polynomials are, up to standardization of the orthogonality interval to $[-1, 1]$, the monic Chebyshev polynomials of the second kind $u_n(x)=2^{-n}U_n(x)$, for which $b=0$ and $a^2=1/4$. This fundamental family is a particular case

($\alpha = \beta = 1/2$) of the Jacobi family $P_n^{(\alpha,\beta)}(x)$, which are orthogonal polynomials with respect to the weight function $(1-x)^\alpha (1+x)^\beta$ on $[-1, 1]$. Closely related to the family $U_n(x)$ are the Chebyshev polynomials of the first kind $T_n(x)$ ($\alpha = \beta = -1/2$), of the third kind $V_n(x)$ ($\alpha = 1/2, \beta = -1/2$) and of the fourth kind $W_n(x)$ ($\alpha = -1/2, \beta = 1/2$). These monic Chebyshev polynomials satisfy the same recurrence relation

$$P_{n+1}(x) = xP_n(x) - \tfrac{1}{4}P_{n-1}(x), \qquad n \geqslant 2, \tag{2.2}$$

but with different initial conditions for P_1 and P_2. These initial conditions are hidden in the Jacobi matrices J_T, J_V, J_W for respectively the polynomials T_n, V_n and W_n, which are explicitly

$$J_T = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & J_U \end{pmatrix}, \qquad J_V = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{4} & J_U \end{pmatrix}, \qquad J_W = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & J_U \end{pmatrix},$$

where J_U is the Jacobi matrix for the Chebyshev polynomials of the second kind U_n. These four families can be embedded in the more general two-parameter family represented by the Jacobi matrix

$$J^{(-1)} = \begin{pmatrix} b & 1 \\ c & J_U \end{pmatrix},$$

and by definition the corresponding monic polynomials P_n satisfy the recurrence relation (2.2) with initial conditions $P_1(x) = x - b$, $P_2(x) = xP_1(x) - cP_0(x)$. All the associated polynomials for $r \geqslant 1$ are the same:

$$P_n^{(r)}(x) = u_n(x), \qquad n \geqslant 0, r \geqslant 1.$$

3. GROSJEAN POLYNOMIALS

The families $\{t_n, n \geqslant 0\}$ and $\{u_n, n \geqslant 0\}$ of Chebyshev polynomials have an interesting extension given by Grosjean. Let us consider first the monic Jacobi family for which the parameters satisfy $\alpha + \beta = -1$, and define $G_0^\alpha = 1$,

$$G_n^\alpha(x) = c_n P_n^{(\alpha, -1-\alpha)}(x), \qquad -1 < \alpha < 0,$$

where c_n is a constant making this a monic polynomial, i.e., $c_n = 2^n / \binom{2n-1}{n}$. These are monic orthogonal polynomials with respect to the weight function

$$w_G(x) = \frac{\sin(-\pi\alpha)}{\pi} \left(\frac{1-x}{1+x}\right)^\alpha \frac{1}{1+x}, \qquad -1 < x < 1. \tag{3.1}$$

A second family of monic Jacobi polynomials with $\alpha+\beta=1$ is denoted by

$$g_n^\alpha(x)=d_n P_n^{(\alpha,\,1-\alpha)}(x), \qquad -1<\alpha<2,$$

with $d_n=2^n\binom{2n+1}{n}^{-1}$, which are monic orthogonal polynomials with weight function

$$w_g(x)=\frac{\sin \pi\alpha}{\alpha(1-\alpha)\pi}\left(\frac{1-x}{1+x}\right)^\alpha (1+x), \qquad -1<x<1. \tag{3.2}$$

Obviously we have the special cases $G_n^{-1/2}(x)=t_n(x)$ and $g_n^{1/2}(x)=u_n(x)$. We will refer to the polynomials G_n^α and g_n^α as the *Grosjean polynomials* of the first and second kind respectively. This terminology is justified since Grosjean [9] showed, using direct verification, that

$$(G_n^\alpha)^{(1)}=g_n^{-\alpha}, \tag{3.3}$$

and Ronveaux [19] gave the generalization of the differential link $t_n'(x)=nu_{n-1}(x)$ between the Chebyshev polynomials

$$(G_n^\alpha)'\,(x)=(-1)^{n-1}\,ng_{n-1}^{-\alpha}(-x)=ng_{n-1}^{1+\alpha}. \tag{3.4}$$

Recall that for $\alpha=-1/2$ we have $u_n(-x)=(-1)^n\,u_n(x)$. Grosjean actually shows [9, p. 275] that the only Jacobi polynomials for which the associated polynomials are again Jacobi polynomials are the Chebyshev polynomials of the first, second, third and fourth kind, and the Grosjean polynomials. Property (3.3) can easily be checked from the recurrence coefficients, which are

$$b_n=\frac{2\alpha+1}{4n^2-1}, \qquad a_n^2=\frac{(n+\alpha)(n-1-\alpha)}{(2n-1)^2}, \qquad \text{for } G_n^\alpha, \tag{3.5a}$$

and

$$b_n=\frac{-2\alpha+1}{4(n+1)^2-1}, \qquad a_n^2=\frac{(n+\alpha)(n+1-\alpha)}{(2n+1)^2}, \qquad \text{for } g_n^\alpha, \tag{3.5b}$$

(see, e.g., Chihara [2, p. 220]) by changing n to $n+1$ and α to $-\alpha$ in (3.5a), which gives (3.5b). Property (3.4) follows by the differential property $(\hat{P}_n^{(\alpha,\beta)})'\,(x)=n\hat{P}_{n-1}^{(\alpha+1,\,\beta+1)}(x)$ for the monic Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}$ and the symmetry property $p_n^{(\alpha,\beta)}(-x)=(-1)^n\,p_n^{(\beta,\alpha)}(x)$ (see, e.g., Szegő [22, p. 63]). It is already interesting to note that for the Grosjean polynomials the a_n^2 are rational function of n consisting of the ratio of two *quadratic* polynomials in n (as in the Legendre case), whereas in general for the Gegenbauer family one has a ratio of two cubic polynomials in n and for the Jacobi polynomials one deals with quartic polynomials in the degree n.

The pair of Grosjean polynomials also give an answer to the following question. Let D denote differentiation $D=d/dx$ and let $L_2=\sigma(x)\,D^2+\tau(x)\,D+\lambda_n$ be the second order (hypergeometric or degenerate hypergeometric) differential operator for the classical orthogonal polynomials (Jacobi, Laguerre, Hermite, Bessel), and $\{P_n,\ n\geqslant 0\}$ be the corresponding family of orthogonal polynomials, where P_n corresponds with the eigenvalue $\lambda_n=-n[(n-1)\,\sigma''+2\tau']/2$, so that $L_2(P_n)=0$. If L_2^* is the formal adjoint of L_2,

$$L_2^*=L_2+2[\sigma'(x)-\tau(x)]\,D+\sigma''-\tau',$$

then for which *orthogonal* family of polynomials $\{P_n^*,\ n\geqslant 0\}$ does one have $L_2^*(P_n^*)=0$? Of course, the Legendre polynomials ($\sigma'=\tau$) solve this problem since then $L_2=L_2^*$. For Grosjean polynomials the operator L_2 is

$$L_{G,\alpha,n}=(1-x^2)\,D^2+(-1-2\alpha-x)\,D+n^2, \qquad L_{G,\alpha,n}(G_n^\alpha)=0,$$

and

$$L_{g,\alpha,n}=(1-x^2)\,D^2+(1-2\alpha-3x)\,D+n(n+2), \qquad L_{g,\alpha,n}(g_n^\alpha)=0,$$

and we have

$$L^*_{G,\alpha,n}=L_{g,-\alpha,n-1},$$

Therefore, if $P_n=G_n^\alpha$, then $P_n^*=g_{n-1}^{-\alpha}$. From (3.4) we also have

$$DL_{G,\alpha,n}=L_{g,1+\alpha,n-1}D.$$

The relative position of the zeros $x_{1,n}^\alpha<x_{2,n}^\alpha<\cdots<x_{n,n}^\alpha$ of G_n^α compared to the zeros $x_{j,n}=-\cos(2j-1)\pi/2n$ of T_n is controlled by a classical comparison theorem due to Markov [22, Theorem 6.12.2]. The ratio between the Chebyshev weight

$$w_T(x)=\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \qquad -1<x<1,$$

and the weight w_G for the Grosjean polynomials G_n^α given by (3.1) is

$$\frac{w_T(x)}{w_G(x)}=\text{const}\times\left(\frac{1+x}{1-x}\right)^{\alpha+1/2}.$$

Now $(1+x)/(1-x)$ is an increasing function of x on the interval $[-1, 1]$, hence for $-1/2<\alpha<0$ the ratio w_T/w_G is an increasing function on $[-1, 1]$, and consequently we have the following inequalities

$$x^{\alpha}_{j,n} < -\cos\frac{(2j-1)\pi}{2n}, \qquad j=1, 2, ..., n, \quad -1/2<\alpha<0.$$

For $-1<\alpha<-1/2$ the ratio w_T/w_G is decreasing, and thus the inequalities for the zeros are reversed

$$x^{\alpha}_{j,n} > -\cos\frac{(2j-1)\pi}{2n}, \qquad j=1, 2, ..., n, \quad -1<\alpha<-1/2.$$

Similar conclusions can be made for the zeros $y^{\alpha}_{j,n}$ $(j=1, 2, ..., n)$ of Grosjean polynomials g^{α}_n of the second kind as compared to the zeros $-\cos(j\pi/n+1)$ $(j=1, 2, ..., n)$ of the Chebyshev polynomials of the second kind. The ratio of the two weights is

$$\frac{w_U(x)}{w_g(x)} = \text{const} \times \left(\frac{1+x}{1-x}\right)^{\alpha-1/2},$$

and thus for $1/2<\alpha<2$ this ratio is an increasing function so that

$$y^{\alpha}_{j,n} < -\cos\frac{j\pi}{n+1}, \qquad j=1, 2, ..., n, \quad 1/2<\alpha<2.$$

For $-1<\alpha<1/2$ the inequalities are reversed

$$y^{\alpha}_{j,n} > -\cos\frac{j\pi}{n+1}, \qquad j=1, 2, ..., n, \quad -1<\alpha<1/2.$$

Finally we note that when we are dealing with Grosjean polynomials of the first kind, the product of the weight function of the orthogonal polynomials and the weight function of the associated polynomials is constant, as in the case of Chebyshev polynomials of the first kind. This is not true when dealing with the Chebyshev polynomials of the third and fourth kind.

4. ANTI-ASSOCIATED ORTHOGONAL POLYNOMIALS

The situation described in Section 2 suggests to construct new families of orthogonal polynomials, which we will denote by $P^{(-r)}_{n+r}$, obtained by pushing down a given Jacobi matrix and by introducing in the empty upper left corner new coefficients b_{-i} $(i=r, r-1, ..., 1)$ on the diagonal and new coefficients $a_{-i}^2 \neq 0$ $(i=r-1, r-2, ..., 0)$ on the lower subdiagonal. The new Jacobi matrix is then of the form

$$J^{(-r)} = \begin{pmatrix} b_{-r} & 1 & & & & & \\ a_{-r+1}^2 & b_{-r+1} & 1 & & & & \\ & a_{-r+2}^2 & b_{-r+2} & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_0^2 & \multicolumn{1}{|c}{b_0} & 1 & \\ & & & & \multicolumn{1}{|c}{a_1^2} & b_1 & 1 \\ & & & & \multicolumn{1}{|c}{} & \ddots & \ddots & \ddots \end{pmatrix}.$$

We will call the orthogonal polynomials $P_n^{(-r)}$ for this Jacobi matrix $J^{(-r)}$ anti-associated polynomials for the family P_n. They contain $2r$ new parameters and satisfy

$$[P_{n+r}^{(-r)}(x)]^{(k)} = P_{n+r}^{(k-r)}(x).$$

For $r=1$ we have

$$J^{(-1)} = \begin{pmatrix} b_{-1} & 1 \\ a_0^2 & J \end{pmatrix},$$

and for $r=2$ we have

$$J^{(-2)} = \begin{pmatrix} b_{-2} & 1 & 0 \\ a_{-1}^2 & b_{-1} & 1 \\ 0 & a_0^2 & J \end{pmatrix}.$$

This new family of anti-associated polynomials $P_n^{(-r)}$ can easily be represented as a combination of the original family P_n and the associated polynomials $P_{n-1}^{(1)}$. First, denote by Q_n the orthogonal polynomials for the finite Jacobi matrix

$$\begin{pmatrix} b_{-r} & 1 & & & \\ a_{-r+1}^2 & b_{-r+1} & 1 & & \\ & a_{-r+2} & b_{-r+2} & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & a_{-1}^2 & b_{-1} \end{pmatrix},$$

so that they satisfy $Q_0=1$, $Q_{-1}=0$ and

$$Q_{n+1}(x)=(x-b_{-r+n})\,Q_n(x)-a^2_{-r+n}Q_{n-1}(x), \qquad n\leqslant r-1.$$

Then, clearly

$$P_n^{(-r)}(x)=Q_n(x), \qquad 0\leqslant n\leqslant r.$$

For $n>r$ the anti-associated polynomials satisfy the three-term recurrence relation

$$P^{(-r)}_{n+r+1}(x)=(x-b_n)\,P^{(-r)}_{n+r}(x)-a_n^2P^{(-r)}_{n+r-1}(x), \qquad n\geqslant 0,$$

so that $P^{(-r)}_{n+r}(x)$ is a solution of the three-term recurrence relation of the original family $P_n(x)$. The initial conditions however are $P_r^{(-r)}(x)=Q_r(x)$ and $P^{(-r)}_{r-1}(x)=Q_{r-1}(x)$, and since every solution of the three-term recurrence relation (2.1) is a linear combination of $P_n(x)$ and $P^{(1)}_{n-1}(x)$, we have

$$P^{(-r)}_{n+r}(x)=AP_n(x)+BP^{(1)}_{n-1}(x), \qquad n\geqslant 0.$$

Using the initial conditions for $n=0$ and $n=1$ gives $A=Q_r(x)$ and $B=-a_0^2Q_{r-1}(x)$, and thus we have

$$P^{(-r)}_{n+r}(x)=Q_r(x)\,P_n(x)-a_0^2Q_{r-1}(x)\,P^{(1)}_{n-1}(x), \qquad n\geqslant 0. \tag{4.1}$$

From this representation it is easy to construct a differential equation satisfied by the family $P^{(-r)}_{n+r}$ if the original family P_n is itself solution of a linear differential equation of second order, for instance when P_n are the classical polynomials (Jacobi, Laguerre, Hermite, Bessel), then they are a solution of the hypergeometric differential equation

$$L_2y\equiv\sigma(x)\,y''+\tau(x)\,y'+\left(\frac{-n}{2}(n-1)\,\sigma''-\tau'n\right)y=0. \tag{4.2}$$

The techniques used in [17–20] and the fact that

$$L_2^*P^{(1)}_{n-1}=(\sigma''-2\tau')\,P_n' \tag{4.3}$$

easily give a fourth order differential equation satisfied by the anti-associated polynomials $P^{(-r)}_{n+r}(x)$. The general technique is the following. Let $B(x)=-a_0^2Q_{r-1}(x)$ and put $J(x)=B(x)\,P^{(1)}_{n-1}(x)$. Then we can transform the equation (4.3) to

$$L_2^*\frac{J}{B}=(\sigma''-2\tau')\,P_n',$$

and introducing the differential operator

$$R_2 = \sigma B^2 D^2 + [(2\sigma' - \tau) B^2 - 2\sigma B B'] D + 2\sigma (B')^2 - \sigma B B''$$
$$- B B' (2\sigma' - \tau) - \left[\frac{\sigma''}{2} (n^2 - n - 2) + \tau' (1 + n) \right] B^2$$

this is equivalent to

$$R_2 J = (\sigma'' - 2\tau') B^3 P_n'. \tag{4.4}$$

From (4.1) we then have

$$R_2 P_{n+r}^{(-r)} = R_2 [Q_r P_n] - (\sigma'' - 2\tau') a_0^6 Q_{r-1}^3 P_n',$$

and eliminating the second derivative using (4.2) then leads to

$$R_2 P_{n+r}^{(-r)} = M_0 P_n + N_0 P_n', \tag{4.5}$$

where M_0 and N_0 are polynomials. Taking the derivative in (4.5) and using (4.2) to eliminate P_n'' also gives

$$\sigma [R_2 P_{n+r}^{(-r)}]' = M_1 P_n + N_1 P_n',$$

where M_1 and N_1 are polynomials, and repeating this also gives

$$\sigma [\sigma [R_2 P_{n+r}^{(-r)}]']' = M_2 P_n + N_2 P_n'.$$

This shows that

$$\det \begin{pmatrix} R_2 P_{n+r}^{(-r)} & M_0 & N_0 \\ \sigma [R_2 P_{n+r}^{(-r)}]' & M_1 & N_1 \\ \sigma [\sigma [R_2 P_{n+r}^{(-r)}]']' & M_2 & N_2 \end{pmatrix} = 0, \tag{4.6}$$

which is the desired fourth order differential equation. When $P_n(x) = G_n^\alpha(x)$, then $\sigma'' - 2\tau' = 0$ so that (4.4) simplifies and becomes homogeneous. The differential equation (4.6) however remains one of the fourth order, except when $\alpha = -1/2$, because then

$$P_{n+r}^{(-r)}(x) = A_0(x) T_n(x) + B_0(x) T_n'(x),$$

where A_0 and B_0 are polynomials. Similar as in the above reasoning we then get a second order differential equation

$$\det \begin{pmatrix} P_{n+r}^{(-r)} & A_0 & B_0 \\ \sigma [P_{n+r}^{(-r)}]' & A_1 & B_1 \\ \sigma [\sigma [P_{n+r}^{(-r)}]']' & A_2 & B_2 \end{pmatrix} = 0,$$

which is equivalent to the equation given in [21].

5. CONSTRUCTION OF THE ORTHOGONALITY MEASURE

If we use probability measures throughout the analysis, and if we use lower case p and q for the orthonormal polynomial, then we have $p_n = \gamma_n P_n$, where

$$\gamma_n = (a_1 a_2 \cdots a_n)^{-1}.$$

Similarly

$$q_n(x) = \frac{Q_n(x)}{a_{-r+1} a_{-r+2} \cdots a_{-r+n}}, \qquad n \leqslant r,$$

and $p_{n-1}^{(1)} = \gamma_n^{(1)} P_{n-1}^{(1)}$ where $\gamma_n^{(1)} = a_1 \gamma_n$. Thus the orthonormal anti-associated polynomials are

$$p_{n+r}^{(-r)}(x) = (a_{-r+1} \cdots a_0)^{-1} \gamma_n P_{n+r}^{(-r)}(x),$$

and using this in (4.1) gives

$$p_{n+r}^{(-r)}(x) = q_r(x) \, p_n(x) - \frac{a_0}{a_1} q_{r-1}(x) \, p_{n-1}^{(1)}(x). \tag{5.1}$$

The orthonormal polynomials are useful in obtaining the weight function for the anti-associated polynomials. Indeed, we can compute the weight function using Christoffel functions

$$\lambda_n(x) = \left(\sum_{j=0}^{n} p_j^2(x) \right)^{-1}$$

by means of the following result of Máté, Nevai and Totik [12, Theorem 8, p. 449]

THEOREM MNT. *Suppose $p_n(x)$ are orthonormal polynomials for a measure μ on $[-1, 1]$ and let*

$$\int_{-1}^{1} \log \mu'(x) \, dx > -\infty.$$

Then

$$\lim_{n \to \infty} n \lambda_n(x) = \pi \mu'(x) \sqrt{1 - x^2} \tag{5.2}$$

holds almost everywhere on $[-1, 1]$.

The above logarithmic condition can be relaxed and in fact it suffices to assume that μ is a regular measure on $[-1, 1]$ (i.e., $\operatorname{supp}(\mu) = [-1, 1]$ and $\lim_{n \to \infty} \gamma_n^{1/n} = 2$) and

$$\int_a^b \log \mu'(x) \, dx > -\infty$$

in order that (5.2) holds almost everywhere on $[a, b] \subset (-1, 1)$ [12, Thm. 8]. We will assume these conditions for the measure μ and moreover we allow the addition of a finite number of mass points to μ. Then (5.2) will still hold almost everywhere on $[a, b]$. Indeed, if we add a mass point c to μ then by the extremum property

$$\lambda_n(x; \mu) = \min_{q_n(x)=1} \int q_n^2(t) \, d\mu(t),$$

where the minimum is taken over all polynomials q_n of degree at most n which take the value 1 at the point x, we see that for the measure $\mu_c = \mu + \varepsilon \delta_c$ (i.e., the measure μ to which we add a mass point at c with mass ε)

$$\begin{aligned} \lambda_n(x; \mu_c) &= \min_{q_n(x)=1} \left(\int q_n^2(t) \, d\mu(t) + \varepsilon q_n^2(c) \right) \\ &\geqslant \min_{q_n(x)=1} \int q_n^2(t) \, d\mu(t) \\ &= \lambda_n(x; \mu), \end{aligned}$$

so that $\liminf_{n \to \infty} n\lambda_n(x; \mu) \geqslant \pi \mu'(x) \sqrt{1-x^2}$, almost everywhere on $[a, b]$. On the other hand, consider the polynomial $q_n(t) = (t-c) \, r_{n-1}(t)/(x-c)$, where r_{n-1} is the minimizing polynomial for the measure $dv(t) = (t-c)^2 \, d\mu(t)$, then for $x \neq c$

$$\begin{aligned} \lambda_n(x; \mu_c) &\leqslant \frac{1}{(x-c)^2} \int (t-c)^2 \, r_{n-1}^2(t) \, d\mu(t) \\ &= (x-c)^{-2} \int r_{n-1}^2(t) \, dv(t) \\ &= (x-c)^{-2} \, \lambda_{n-1}(x; v) \end{aligned}$$

so that $\limsup_{n\to\infty} n\lambda_n(x;\mu_c) \leqslant \lim_{n\to\infty}(x-c)^{-2}\, n\lambda_n(x;\nu) = \pi\mu'(x)\sqrt{1-x^2}$ almost everywhere on $[a, b]$, since ν is regular and satisfies the Szegő condition on $[a, b]$. Combined with the previous inequality this gives

$$\lim_{n\to\infty} n\lambda_n(x;\mu_c) = \pi\mu'(x)\sqrt{1-x^2}$$

almost everywhere on $[a, b]$, independent of the mass point c. This procedure can be repeated, so that adding a finite number of mass points does not change the behavior in (5.2).

In the previous discussion we wanted to allow the measure μ to have a larger support than $[-1, 1]$ (by allowing mass points outside $[-1, 1]$) but in such a way that (5.2) still holds. Alternatively we can use a weaker result by Nevai [13, Thm. 54, p. 104] in which the support of μ is allowed to be $[-1, 1] \cup E$, where E contains at most a denumerable number of points which can only accumulate at ± 1.

THEOREM N. *Suppose $\mu \in M(0, 1)$, i.e., the recurrence coefficients a_n and b_n have asymptotic behavior given by*

$$\lim_{n\to\infty} a_n = 1/2, \qquad \lim_{n\to\infty} b_n = 0.$$

Then

$$\limsup_{n\to\infty} n\lambda_n(x;\mu) = \pi\mu'(x)\sqrt{1-x^2}$$

holds for almost every $x \in \operatorname{supp}(\mu)$.

With this result, if we can compute the limit of $n\lambda_n(x;\mu)$ for $x \in (-1, 1)$, then this limit is almost everywhere equal to $\pi\mu'(x)\sqrt{1-x^2}$, which allows us to compute the weight μ' on $(-1, 1)$ without having to worry about the fact that $\operatorname{supp}(\mu)$ may be larger than $[-1, 1]$. The existence of such a limit is however not guaranteed by Nevai's theorem, contrary to the theorem of Máté, Nevai and Totik where the existence of the limit is in the conclusion of the theorem but where $\operatorname{supp}(\mu) = [-1, 1]$ is required.

If we assume that the original family p_n belongs to a measure μ in the class $M(0, 1)$, then the anti-associated polynomials $p_n^{(-r)}$ will also have a measure $\mu^{(-r)}$ which belongs to the class $M(0, 1)$, and thus we know that $\mu^{(-r)}$ has support $[-1, 1] \cup E$, where E is at most denumerable with the only accumulation points at ± 1 [24, Thm. 1, p. 437]. In fact, there can be at most $2r$ mass points, since the associated polynomials of order r of the anti-associated polynomials $p_n^{(-r)}$ are again p_n and these have all their zeros inside $[-1, 1]$, and the interlacing property of orthogonal polynomials and associated orthogonal polynomials shows that adding one row and one column in the Jacobi matrix to form the anti-associated polynomials $p_n^{(-1)}$ can at most add a mass point to the left of -1 and to the right of 1. Adding r rows and columns thus can add at most r mass points to the left of -1 and r mass points to the right of 1. If the original measure μ belongs to the Szegő class (which is a subclass of $M(0,1)$), then the measure $\mu^{(-r)}$ restricted to $[-1,1]$ also belongs to the Szegő class. This is so because if we denote by $\mu_{[-1,1]}^{(-r)}$ the restriction of $\mu^{(-r)}$ to the interval $[-1,1]$, then $\lim_{n\to\infty} \gamma_n(\mu^{(-r)})/\gamma_n(\mu_{[-1,1]}^{(-r)})$ exists and is strictly positive [13, Thm. 25 on p. 136] and since $\gamma_n(\mu^{(-r)}) = \gamma_n(\mu)(a_{-r+1}\cdots a_0)^{-1}$ and μ belongs to the Szegő class, which is equivalent with the statement that $\lim_{n\to\infty} \gamma_n/2^n$ exists and is strictly positive, it follows that $\lim_{n\to\infty} \gamma_n(\mu_{[-1,1]}^{(-r)})/2^n$ exists and is strictly positive, so that $\mu_{[-1,1]}^{(-r)}$ belongs to the Szegő class.

In order to use the result in (5.2) we observe that for $n \geqslant r$

$$\sum_{j=0}^{n} [p_j^{(-r)}(x)]^2 = \sum_{j=0}^{r-1} q_j^2(x) + \sum_{j=0}^{n-r} [p_{j+r}^{(-r)}(x)]^2,$$

and using (5.1) this gives

$$\sum_{j=0}^{n-r} [p_{j+r}^{(-r)}(x)]^2 = q_r^2(x) \sum_{k=0}^{n-r} p_k^2(x) + \left(\frac{a_0}{a_1}\right)^2 q_{r-1}^2(x) \sum_{k=0}^{n-r-1} [p_k^{(1)}(x)]^2$$
$$-\frac{2a_0}{a_1} q_r(x)\, q_{r-1}(x) \sum_{k=1}^{n-r} p_k(x)\, p_{k-1}^{(1)}(x).$$

In order to be able to compute these sums, we will need to be able to compute Christoffel functions of associated polynomials and sums of mixed form containing the product $p_n(x)\, p_{n-1}^{(1)}(x)$. This is in general not so easy. However, when we use Grosjean polynomials of the first kind, then the associated polynomials are Grosjean polynomials of the second kind, so that we are always dealing with Jacobi polynomials, for which we can compute these sums, at least as n becomes large. For the anti-associated polynomials corresponding to Grosjean polynomials we thus can prove the following result.

THEOREM 1. *If the original system of orthogonal polynomials consists of Grosjean polynomials of the first kind, then the anti-associated polynomials* $p_n^{(-r)}$ *are orthogonal with respect to a measure* $\mu^{(-r)}$ *which is absolutely continuous on* $[-1,1]$ *with density*

$$w_r(x) = \frac{\sin(-\pi\alpha)}{\pi} \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \left| q_r(x) - a_0 e^{i\alpha\pi} q_{r-1}(x) \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \right|^{-2}.$$

In addition, there may be at most 2r mass point outside $[-1, 1]$ *(r to the left of* -1 *and r to the right of* 1*). These mass points are roots of the equation*

$$q_r(x) - a_0 q_{r-1}(x) \operatorname{sign} x \frac{|x-1|^\alpha}{|x+1|^{\alpha+1}} = 0.$$

Proof. If the original family consists of Grosjean polynomials, then by (4.3) and (3.1) we have for $-1 < x < 1$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-r} p_k^2(x) = \frac{1}{\sqrt{1-x^2}} \frac{1}{\sin(-\pi\alpha)} \frac{(1+x)^{\alpha+1}}{(1-x)^\alpha}, \tag{5.3}$$

and this even holds uniformly on closed subintervals of $(-1, 1)$, and similarly by using (3.2) (with α replaced by $-\alpha$)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-r-1} [p_k^{(1)}(x)]^2 = \frac{1}{\sqrt{1-x^2}} \frac{-2\alpha(1+\alpha)}{\sin(-\pi\alpha)} \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}}. \tag{5.4}$$

For the remaining sum, which consists of a mixture of the original polynomials and the associated polynomials, we can use Darboux's extension of the Laplace–Heine formula for Legendre polynomials [22, Thm., 8.21.8 on p. 196]

$$\sqrt{n\pi}\, P_n^{(\alpha,\beta)}(\cos\theta) = \left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{-\beta-1/2} \times \cos\left([n + (\alpha+\beta+1)/2]\,\theta - \frac{\alpha+1/2}{2}\pi\right) + O(1/n), \tag{5.5}$$

which holds uniformly for $x = \cos\theta$ on closed intervals of $(-1, 1)$. For Grosjean polynomials $P_n(x) = G_n^\alpha(x)$ we have

$$p_n(x) = [1 + O(1/n)] \sqrt{\frac{2n\pi}{\sin(-\pi\alpha)}}\, P_n^{(\alpha,\,-1-\alpha)}(x),$$

$$p_{n-1}^{(1)}(x) = [1 + O(1/n)] \sqrt{\frac{-n\pi\alpha(1+\alpha)}{\sin(-\pi\alpha)}}\, P_{n-1}^{(-\alpha,\,1+\alpha)}(x),$$

and hence for $x = \cos\theta$ with $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$

$$p_n(x) = \sqrt{\frac{2}{\sin(-\pi\alpha)}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{\alpha+1/2} \times \cos\left(n\theta - \frac{\alpha+1/2}{2}\pi\right) + O(1/n)$$

and

$$p_{n-1}^{(1)}(x) = \sqrt{\frac{-\alpha(1+\alpha)}{\sin(-\pi\alpha)}} \left(\sin\frac{\theta}{2}\right)^{\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{-\alpha-3/2} \times \cos\left(n\theta + \frac{\alpha-1/2}{2}\pi\right) + O(1/n).$$

Using this gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-r} p_k(x)\, p_{k-1}^{(1)}(x) = \frac{\sqrt{-2\alpha(1+\alpha)}}{\sin(-\pi\alpha)} \frac{1}{\sin\theta/2\cos\theta/2} \times \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-r} \cos\left(k\theta - \frac{\alpha+1/2}{2}\pi\right) \cos\left(k\theta - \frac{-\alpha+1/2}{2}\pi\right).$$

Using $2\cos a\cos b = \cos(a+b) + \cos(a-b)$ gives

$$\frac{1}{n} \sum_{k=1}^{n-r} \cos\left(k\theta - \frac{\alpha+1/2}{2}\pi\right) \cos\left(k\theta - \frac{-\alpha+1/2}{2}\pi\right) = \frac{1}{2n} \sum_{k=1}^{n-r} [\cos(2k\theta - \pi/2) + \cos\alpha\pi]$$

and thus uniformly for x on closed subsets of $(-1, 1)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-r} p_k(x)\, p_{k-1}^{(1)}(x) = \frac{\sqrt{-2\alpha(1+\alpha)}\cos\pi\alpha}{\sin\theta\sin(-\pi\alpha)}. \tag{5.6}$$

Alternatively, (5.5) can also be obtained by using a Christoffel–Darboux type formula [1, corollary 2.12] which for orthonormal polynomials is

$$\sum_{k=1}^{n} p_k(x)\, p_{k-1}^{(1)}(x) = a_{n+1}[p'_{n+1}(x)\, p_{n-1}^{(1)}(x) - p'_n(x)\, p_n^{(1)}(x)].$$

When the original system consists of Grosjean polynomials of the first kind G_n^α one knows by (3.3) that the associated polynomials are Grosjean polynomials of the second kind $g_n^{-\alpha}$ and by (3.4) we know that the derivative of G_n^α is $ng_{n-1}^{1+\alpha}$. For the orthonormal polynomials this gives

$$p_n(x) = [1+O(1/n)] \sqrt{\frac{2n\pi}{\sin(-\pi\alpha)}}\, P_n^{(\alpha,\,-1-\alpha)}(x),$$

$$p_n^{(1)}(x) = [1+O(1/n)] \sqrt{\frac{-n\pi\alpha(1+\alpha)}{\sin(-\pi\alpha)}}\, P_n^{(-\alpha,\,1+\alpha)}(x),$$

$$p_n'(x) = [1+O(1/n)] \sqrt{\frac{2\pi n}{\sin(-\pi\alpha)}}\, \frac{n}{2}\, P_{n-1}^{(1+\alpha,\,-\alpha)}(x),$$

and hence using (5.5) gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_k(x)\, p_{k-1}^{(1)}(x)$$

$$= \frac{\sqrt{-2\alpha(1+\alpha)}}{\sin(-\pi\alpha)\sin^2\theta} \lim_{n\to\infty} \left[\cos\left((n+1)\theta - \frac{\alpha+3/2}{2}\pi\right) \cos\left(n\theta - \frac{-\alpha+3/2}{2}\pi\right) \right.$$

$$\left. -\cos\left(n\theta - \frac{\alpha+3/2}{2}\pi\right) \cos\left((n+1)\,\theta - \frac{-\alpha+1/2}{2}\pi\right)\right]$$

Simple trigonometry then gives (5.6). Combining the limiting relations (5.3), (5.4), and (5.6) gives uniformly for x on closed intervals of $(-1, 1)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} [p_j^{(-r)}(x)]^2 = \frac{1}{\sqrt{1-x^2}} \frac{1}{\sin(-\pi\alpha)} \left[q_r^2(x) \frac{(1+x)^{\alpha+1}}{(1-x)^\alpha} \right.$$

$$-2\frac{a_0}{a_1} q_r(x)\, q_{r-1}(x) \cos\alpha\pi \sqrt{-2\alpha(1+\alpha)}$$

$$\left. +\left(\frac{a_0}{a_1}\right)^2 q_{r-1}^2(x)(-2\alpha)(1+\alpha) \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \right]$$

Observe now that $a_1 = \sqrt{-2\alpha(1+\alpha)}$, thus we find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} [p_j^{(-r)}(x)]^2$$

$$= \frac{1}{\sin(-\pi\alpha)} \frac{1}{\sqrt{1-x^2}} \frac{(1+x)^{\alpha+1}}{(1-x)^\alpha} \left| q_r(x) - a_0 e^{i\alpha\pi} q_{r-1}(x) \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \right|^2,$$

and this holds uniformly on closed intervals $[a, b] \subset (-1, 1)$. From (5.2) we can then conclude that the orthogonality measure $\mu^{(-r)}$ for the anti-associated polynomials has an absolutely continuous part on $(-1, 1)$ with weight function

$$w_r(x) = \frac{\sin(-\pi\alpha)}{\pi} \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \left| q_r(x) - a_0 e^{i\alpha\pi} q_{r-1}(x) \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}} \right|^{-2}, \quad (5.7)$$

which is the weight function of the original system p_n (Grosjean polynomials of the first kind) divided by a positive factor containing the new parameters $b_{-r}, ..., b_{-1}$ and $a^2_{-r+1}, ..., a^2_{-1}, a^2_0$. This factor cannot vanish on $(-1, 1)$, because then both the real part and the imaginary part of

$$q_r(x) - a_0 e^{i\alpha\pi} q_{r-1}(x) \frac{(1-x)^\alpha}{(1+x)^{\alpha+1}}$$

need to vanish. The imaginary part can only vanish when $q_{r-1}(x) = 0$, and assuming this, the real part can then only vanish when also $q_r(x) = 0$. This is impossible since two consecutive orthogonal polynomials cannot have a common zero.

Since the original orthogonal polynomials are Jacobi polynomials, they will belong to the class $M(0, 1)$. Moreover, the original system satisfies

$$\sum_{k=0}^{\infty} (|1 - 4a^2_{k+1}| + 2\,|b_k|) < \infty,$$

and hence also the new system of anti-associated polynomials satisfies this *trace class* condition. But then we know [24, Theorem 6] that the orthogonality measure $\mu^{(-r)}$ is absolutely continuous on $(-1, 1)$ and we have obtained the weight function in (5.7). The mass points outside $(-1, 1)$ are those points $x \in \mathbb{R} \setminus (-1, 1)$ for which $\sum_{k=0}^{\infty} [p_k^{(-r)}(x)]^2 < \infty$. This means that at a mass point x we have $p_n^{(-r)}(x) \to 0$, and from (5.1) this implies that

$$\lim_{n \to \infty} p_n(x) \left[q_r(x) - \frac{a_0}{a_1} q_{r-1}(x) \frac{p^{(1)}_{n-1}(x)}{p_n(x)} \right] = 0.$$

For $x \notin [-1, 1]$ we know that $|p_n(x)|$ increases exponentially fast, hence at a mass point we always have

$$q_r(x) - \frac{a_0}{a_1} q_{r-1}(x) \lim_{n \to \infty} \frac{p^{(1)}_{n-1}(x)}{p_n(x)} = 0.$$

The limit of the ratio $p^{(1)}_{n-1}(x)/p_n(x)$ can be found by using Markov's theorem, which states that this limit is the Stieltjes transform of the measure μ, or it can be obtained from Darboux's generalization of the Laplace–Heine formula for Legendre polynomials for x outside $[-1, 1]$ [22, Theorem 8.21.7]. Both methods give

$$\lim_{n\to\infty} \frac{p^{(1)}_{n-1}(x)}{a_1\, p_n(x)} = \frac{(x-1)^\alpha}{(x+1)^{\alpha+1}},$$

where the right hand side is to be taken positive if $x>1$ and negative if $x<-1$. This means that a mass point satisfies

$$q_r(x) - a_0 q_{r-1}(x)\, \frac{(x-1)^\alpha}{(x+1)^{\alpha+1}} = 0.$$

Clearly, there can be at most $2r$ mass points, as was indicated earlier. ∎

Remark. In order to determine the measure $\mu^{(-r)}$ for the anti-associated polynomials corresponding to Grosjean polynomials, one can also use the results from Theorem 3.9 in Peherstorfer [16] using the Cauchy principal value and the Stieltjes transform. Indeed, the Grosjean polynomials are one of the seldom cases where a nice explicit expression for the Cauchy principal value and the Stieltjes transform exists and for this reason the results of [16] can be applied without problems. Observe that this technique is basically also the one used by Grosjean in [9]. Our approach using Christoffel functions has the advantage that it avoids taking boundary values of a Stieltjes transform or evaluating a Cauchy principal values and uses only information of the orthogonal polynomials on the interval $[-1, 1]$. In particular our method also works when appropriate asymptotic information of the orthogonal polynomials and the associated orthogonal polynomials on the real line is available.

6. EXAMPLES

The simplest examples occur when we take $\alpha = -1/2$, in which case the original system consists of Chebyshev polynomials of the first kind. The weight function for the anti-associated polynomials then becomes

$$\begin{aligned} w_r(x) &= \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \frac{1}{|q_r(x) - i a_0 q_{r-1}/\sqrt{1-x^2}|^2} \\ &= \frac{1}{\pi} \frac{\sqrt{1-x^2}}{(1-x^2)\, q_r^2(x) + a_0^2 q_{r-1}^2(x)}. \end{aligned}$$

Hence this weight function is the weight function of Chebyshev polynomials of the second kind, divided by a polynomial which is positive on $[-1, 1]$. Such orthogonal polynomials are known as Bernstein–Szegő polynomials [22, Section 2.6].

In case the original system consists of Chebyshev polynomials of the second kind, for which all the recurrence coefficients are constant, we have $p_n = U_n$ and $p_{n-1}^{(1)} = U_{n-1}$. Relation (5.1) then becomes

$$p_{n+r}^{(-r)}(x) = q_r(x)\, U_n(x) - 2a_0 q_{r-1}(x)\, U_{n-1}(x).$$

Using $U_n(x) = \sin(n+1)\,\theta / \sin\theta \quad (x = \cos\theta)$, one easily shows for $-1 < x < 1$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} U_j^2(x) = \frac{1}{2(1-x^2)}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U_j(x)\, U_{j-1}(x) = \frac{x}{2(1-x^2)},$$

and hence

$$\frac{1}{n} \sum_{j=0}^{n} [p_j^{(r)}(x)]^2 = \frac{1}{2(1-x^2)} [q_r^2(x) - 4a_0 x q_r(x)\, q_{r-1}(x) + 4a_0^2 q_{r-1}^2(x)],$$

and hence the weight function becomes

$$w(x) = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{q_r^2(x) - 4a_0 x q_r(x)\, q_{r-1}(x) + 4a_0^2 q_{r-1}^2(x)},$$

i.e., this is again a Bernstein–Szegő weight. Observe that it can be written as

$$w(x) = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{|q_r(x) - 2a_0 e^{i\theta} q_{r-1}(x)|^2}.$$

For the mass points we see that they can occur only for $x \notin [-1, 1]$ when

$$q_r(x) - 2a_0 q_{r-1}(x) \lim_{n\to\infty} \frac{U_{n-1}(x)}{U_n(x)} = 0,$$

and since $U_{n-1}(x)/U_n(x) \to 1/(x+\sqrt{x^2-1})$ for $x \notin [-1, 1]$, where the limit is to be taken positive for $x > 1$ and negative for $x < -1$, it follows that x is a mass point only when

$$q_r(x) - 2a_0 q_{r-1}(x) \frac{1}{x+\sqrt{x^2-1}} = 0. \tag{6.1}$$

If we multiply the left hand side by $q_r(x) - 2a_0 q_{r-1}(x)/(x-\sqrt{x^2-1})$, then this implies

$$q_r^2(x) - 4a_0 x q_r(x)\, q_{r-1}(x) + 4a_0^2 q_{r-1}^2(x) = 0.$$

So the mass points are zeros of the polynomial in the denominator of the weight function, but only those zeros for which (6.1) holds.

For $r = 1$ we can consider the following cases:

(1) $a_0 = 1/\sqrt{2}$ and $b_{-1} = 0$. In this case the weight function of the anti-associated polynomials is $w(x) = 1/(\pi\sqrt{1-x^2})$ and thus we have the Chebyshev polynomials of the first kind.

(2) $a_0 = 1/2$ and $b_{-1} = -1/2$. Then the weight function is the one for Jacobi polynomials with $\alpha = 1/2$ and $\beta = -1/2$ and thus we have Chebyshev polynomials of the third kind $V_n(x)$.

(3) Finally when $a_0 = 1/2$ and $b_{-1} = 1/2$ the weight function becomes the one for Jacobi polynomials with $\alpha = -1/2$ and $\beta = 1/2$, so that we have Chebyshev polynomials of the fourth kind $W_n(x)$.

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